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Supersymmetric M-theory compactifications with fluxes on seven-manifolds and G-structures

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Abstract: We consider Minkowski compactifications of M-theory on generic seven-dimensional manifolds. After analyzing the conditions on the four-form flux, we establish a set of relations between the components of the intrinsic torsion of the internal manifold and the components of the four-form flux needed for preserving supersymmetry. The existence of two nowhere vanishing vectors on any seven-manifold with G_2 structure plays a crucial role in our analysis, leading to the possibility of four-dimensional compactifications with $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetry.

Keywords: M-Theory, Supergravity Models.

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1. Introduction

String theory compactifications with background fluxes are an old subject of study, but in spite of their physical interest they are not yet as well understood as their purely geometric counterparts. Different directions of research on the more general task of understanding supersymmetric solutions have recently been focusing on the language and techniques of G-structures. So far this has been applied to Neveu-Schwarz three-form [1]–[4] or to the two-form flux in type IIA [5, 6]. It is obviously interesting to extend these methods to Ramond-Ramond fluxes. In type IIA these different fluxes get organized in terms of the four-form flux of M-theory. In this paper we therefore reconsider compactifications of M-theory to four dimensions in the presence of background four-form fluxes and analyze the conditions under which the vacuum preserves $\mathcal{N}=1$ supersymmetry in four dimensions.[1]

The traditional approach has been so far largely based on Ansätze for the conserved spinor, the metric and the background fluxes [8]–[14]. An incomplete list of closely related M-brane solutions is [15]–[17]. The idea is that G-structures provide instead an organizing principle and help to draw more general conclusions; much as in the purely geometrical case, where possible internal metrics for compactifications have long since been classified using the concept of holonomy. In that case the lack of explicit expressions for the metric has sometimes been largely compensated by the amount of mathematical results known about them, and one can hope that this happy story repeats here to some extent. We

[1] G-structures have also been applied [7] to M-theory to classify all possible supersymmetric solutions in 11 dimensions, without reference to compactifications. Our use of SU(3) structures in seven dimensions somewhat parallels the one in [7] of SU(5) structures in eleven.

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hasten to add that compact nonsingular seven-manifolds are subject here to the usual simple no-go arguments coming from leading terms in the equations of motion [18, 19, 12], which remain untouched by our analysis; we are adding nothing here to the usual strategies to avoid this argument, such as invoking higher-derivatives terms (not fully under control however as of this writing) and/or sources.

In order to preserve any supersymmetry upon compactification, the internal manifold has to carry a spin-structure. Decomposing the supersymmetry generator as a simple product $\epsilon = \psi \otimes \vartheta$, however, one runs into difficulties when trying to satisfy the supersymmetry constraints in the presence of internal background fluxes. Our way to circumvent these problems is to use a more general decomposition of ϵ, that we are allowed to make due to a theorem proven in [20]. This theorem states that on a seven manifold a spin-structure already implies the existence of an SU(2)-structure, and thus also of SU(3)- and G_2-structures. The proof of this theorem uses the existence of two nowhere vanishing linearly independent vector fields on such manifolds. Whereas the implication SU(2) $\to$ SU(3) $\to G_2$ of structures is obvious, these vector fields allow one to construct them in the reversed order. Our starting point will therefore be a G_2-structure with its associated nowhere vanishing real spinor ϑ, which we can take to be of unit norm. Out of this one can build a whole U(1) of SU(3)-structures whose associated normalized spinors are

$$\hat{\vartheta}_{\pm} = \frac{1}{\sqrt{2}} e^{\pm i\xi} \left(1 \pm v^a \gamma_a\right) \vartheta \,, \tag{1.1}$$

where v^a are the components of one of the two vector fields, which are again taken to be of unit norm, and where the phase $e^{i\xi}$ parametrizes the different SU(3)-structures. An ansatz without this phase has already been made in [9].

We can then decompose the supersymmetry generator as

$$\epsilon = \psi_+ \otimes \vartheta_+ + \psi_- \otimes \vartheta_- \,, \tag{1.2}$$

where $\psi_\pm = \psi_\mp^*$ are chiral spinors of opposite chiralities in four dimensions and $\vartheta_\pm$ are the SU(3) invariant spinors (1.1) up to a possible normalization. Imposing the supersymmetry constraints then leads to certain restrictions. These are of two types. On the one hand side there are constraints on the four-form G and the warp factor Δ, in which derivatives are only present in the form $d\Delta$ (and indirectly in the definition of $G = dC$) as will be seen in (3.12a), (3.12b) and (3.21). As a comparison, let us recall that in M-theory compactifications on four-folds [21] there were primitivity constraints on G, whereas for two–form flux on manifolds of SU(3) structure a "holomorphic monopole equation" arose. On the other side there are differential equations involving the tensors (v, J, Ω) that specify the above mentioned SU(3)-structures:

$$\begin{aligned} d(e^{2\Delta} v) &= 0 \,, & d(e^{4\Delta} J) &= -2\, e^{4\Delta} (*G) \,, \\ d(e^{3\Delta} \Omega) &= 0 \,, & d(e^{2\Delta} J \wedge J) &= -6 e^{2\Delta} G \wedge v \,. \end{aligned} \tag{1.3}$$

These equations are very similar to those found for NS three-form [2] or RR two-form [5]. This is no coincidence: the structure of these equations is consistent with a brane interpretation [7]. In particular, J is said to be a *generalized calibration* for a five-brane

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that wraps a two-cycle inside M. This cycle can then be shown to minimize the energy of the brane, which takes into account both the volume and the integral of the flux. In particular there can be a non-trivial minimal energy cycle even in a trivial homology class. This is somewhat in parallel with the fact for example that having an SU(2) structure does not imply that one has a non-trivial four-cycle, as patently recalled by the above mentioned theorem about G-structures on seven-manifolds [20].

In terms of G-structures, equations (1.3) can be interpreted instead as computing intrinsic torsions (as well as determining G from the second one), quantities which measure the extent to which the manifold fails to have G-holonomy — if one prefers, the amount of back-reaction. These objects are used to classify manifolds with G-structures. For instance, a weakly G_2 manifold is in this language simply a manifold with G_2-structure whose intrinsic torsion is in the singlet representation. Conformally G_2-holonomy manifolds have torsion in the vector representation, and so on. In our case, what (1.3) teach us are SU(3) torsions, which although containing more information, seem less useful for classification purposes. For this reason, we computed the G_2 torsions relative to the G_2 structure defined by ϑ above. Although these do not contain all the information about supersymmetry, they give simple necessary criteria for which G_2-structure manifolds can be used in presence of which fluxes. Finally, a point on which our geometrical program fails is that the Bianchi identity (which does not in general follow from supersymmetry) needs to be imposed separately, as indeed it was done in all explicit examples based on Ansätze. In general, the intrinsic torsion can also be shown to satisfy differential equations and it is a priori not inconceivable that one might find cases in which this helps to solve Bianchi, but this will not be settled here.

2. Basics about G-structures

Before starting with our analysis, we recall in this section some basic concepts about G-structures. For details we refer the reader for example to [22].

Consider an n-dimensional manifold Y, its tangent bundle TY and its frame bundle FY. In general the latter has $\mathrm{GL}(n,\mathbb{R})$ as structure group, namely this is the group in which its transition functions take value. It can happen though that we can work with a smaller object: There can be a subbundle of FY, still principal (the structure group acts on the fibers in the adjoint), but whose fibers are isomorphic to a smaller subgroup $G \subset \mathrm{GL}(n,\mathbb{R})$. This subbundle is called a G-structure on the manifold. The existence of such a structure is a topological constraint. It implies that the structure group of TY is reduced to G.

Tensors on Y transform in some representation of the structure group $\mathrm{GL}(n,\mathbb{R})$. If a G-structure reduces this group to $G \subset \mathrm{GL}(n,\mathbb{R})$, then singlets may occur in the decomposition of the $\mathrm{GL}(n,\mathbb{R})$-representation into irreducible G-representations, and these singlets can be used to define alternatively the G-structure.

A prime example is given by a riemannian manifold. Existence of a riemannian metric g on Y allows one to define an $\mathrm{O}(n)$ subbundle of FY defined by frames which are orthonormal, namely in which the metric is written as δ_{ab}. Orientability reduces to $\mathrm{SO}(n)$ and a spin

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structure is a reduction to Spin(n). Now, if $G \subset$ Spin(n), also the spin-representation of SO(n) will contain some singlets under G corresponding to nowhere vanishing G-invariant spinors on Y, that one can choose to have unit norm. So in these cases there is yet another way of characterizing the G-structure, through a G-invariant spinor. Obviously this is the case of interest for supersymmetry. The G-invariant tensors mentioned above can be recovered from this spinor ϑ as bilinears $\vartheta^\dagger \gamma_{i_1 \ldots i_n} \vartheta$. We will make all these concepts more explicit shortly for the case of SU(3) and G_2 structures in seven dimensions.

Obviously if there is a smaller G-structure it implies trivially the existence of a bigger one. Translating this in the tensor language, it is interesting to notice that this allows to recover the tensor characterizing the bigger structure from the tensor characterizing the smaller structure. For example, since $G_2 \subset SO(7)$ the existence of a G_2 invariant three-form allows to find a metric associated to it — this is the well-known formula for g in terms of Φ. To go from a bigger to a smaller G structure is instead not obvious. If we now restrict our attention to dimension seven we however find a surprise. In [23] it had been shown that any compact, orientable seven-manifold admits two linearly independent never vanishing vector fields. This makes use of an index invariant for fields of 2-vectors on n-manifolds analogous to the one used for simple vector fields. But, instead of being defined in $\pi_{n-1}(S^{n-1}) = \mathbb{Z}$, it is defined in $\pi_{n-1}(V_{n,2})$, a homotopy group of a Stiefel manifold. For a compact orientable seven-manifold this index simply happens to vanish. This has been used in [20] to show that[2] *a compact spin seven-manifold admits an* SU(2)*-structure.* As we said, this implies in particular SU(3) and G_2 structure. This simply means that instead of using both the vectors we only use one or none respectively.

We now specialize as promised the above general discussion about G-structures and invariant tensors to the cases relevant to us.

As mentioned above, all these G-structures for $G \subset SO(7)$ come together with certain G-invariant spinors. The G_2 case is by now familiar: the invariant tensor which determines it is a three-form Φ. This satisfies the octonionic structure constants:

$$\Phi_{abe}\Phi^{cd}{}_{e} = 2\delta_{ab}{}^{[cd]} - (*\Phi)_{ab}{}^{cd} \,.$$

Φ singles out one G_2-invariant nowhere vanishing real spinor ϑ, in terms of which $\Phi_{abc} = -i\, \vartheta^\dagger \gamma_{abc} \vartheta$.

An SU(3)-structure in seven dimensions is given by tensors (v, J, ψ_3): a nowhere vanishing (which we take to be normalized to 1) vector field v (we will use the same symbol to denote the vector field and its dual (via the metric) one-form), a generalized almost complex structure J (again the same symbol will denote its associated two-form) and a real three-form ψ_3. In order to define an SU(3)-structure in seven dimensions they furthermore

[2]Note that the authors of [20] call a topological G-structure what we call a G-structure, and a geometric G-structure what we call a torsion-free G-structure. We also note that though the results of [23] and [20] are stated for compact seven-manifolds, the proofs rely largely on the dimensionalities and hold for non-compact case.

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have to satisfy

$$
\begin{aligned}
&(i) \quad v \lrcorner J = 0\,,\\
&(ii) \quad v \lrcorner \psi_3 = 0\,,\\
&(iii) \quad J^a{}_b J^b{}_c = -\delta^a{}_c + v^a v_c\,,\\
&(iv) \quad \psi_3(X, JY, Z) = \psi_3(JX, Y, Z)\,.
\end{aligned}
\tag{2.1}
$$

These relations are loosely speaking a "dimensional reduction" along v of the octonionic structure constants given above. ψ_3 is to be thought of as $\mathrm{Re}\{\Omega\}$. We can determine the two- and three-form in terms of the vector and an underlying G_2-structure as,

$$
J = v \lrcorner \Phi \qquad \text{and} \qquad \psi_3 = \mathrm{Re}\left\{ e^{2i\xi} \left[i\Phi - iv \wedge (v \lrcorner \Phi) - v \lrcorner (*\Phi) \right] \right\}, \tag{2.2}
$$

where the phase $e^{2i\xi}$ is a parameter of the SU(3)-structure.

Again, besides such a description in terms of tensors there is one in terms of spinors $\hat{\vartheta}_\pm$, which is the one which comes naturally out of supersymmetry, as outlined in the introduction. Again the tensors are bilinears of the spinor, as we will see in more detail in what follows. Conversely, given the tensors one gets two SU(3)-invariant spinors on Y. The first one is given by the G_2-invariant spinor ϑ associated to the underlying G_2-structure on Y and the second one by $v \cdot \vartheta \equiv v^a \gamma_a \vartheta$. As usual, ϑ can be taken to be of unit norm in which case $v \cdot \vartheta$ has norm one as well.

Now, if we take linear combinations $\hat{\vartheta}_+ = \frac{1}{\sqrt{2}} e^{i\xi} (1 + v^a \gamma_a)\, \vartheta$ as in (1.1), it is easy to see that forming bilinears one gets back the tensors one started with. One might wonder whether this inverse is unique. A priori other linear combinations might work, but by Fierz identities one can show $v \cdot \hat{\vartheta}_\pm = \pm \hat{\vartheta}_\pm$, from which (1.1) follows. Interestingly, we will find the same result also in another way, while considering the differential equations coming from bilinears in the next section.

We finally also introduce intrinsic torsions, which we will later see practically into play. This comes about while comparing connections on the bundles TY, FY and P, the latter being the principal bundle which defines the G-structure. Connections on TY and FY are in one-to-one correspondence. Any connection on P lifts to a unique connection on FY, whereas a connection on FY reduces to a connection on the subbundle P if and only if its holonomy (and the holonomy of the corresponding connection on TY) is contained in G. Hence connections on P are in one-to-one correspondence with connections of holonomy G on TY. However, there can be an obstruction against finding any connections on P that induce *torsion-free* connections on TY. This obstruction is called the intrinsic torsion of the G-structure P. If it is non-vanishing, then in particular the Levi-Civita connection on TY cannot have holonomy in G and the normalized G-invariant tensors and spinors are not covariantly constant in the Levi-Civita connection. Its definition is given in terms of the torsion of the difference of any two connections on P. All we need is that it is a section of $G^\perp \otimes TY$, where $G^\perp$ is the quotient of $adFY$ by adP. Then we can decompose the tensor product in representations of G and get a certain number of tensors which we can equally well call intrinsic torsion. The prettiest example for seven dimensions is given by

G_2 intrinsic torsion. In this case $G^\perp$ is in the representation **7** of G_2, since the adjoint of SO(7) decomposes under G_2 as **21**= **14** + **7**. Now we get $G_2^\perp \otimes TY$=**7** $\otimes$ **7**= **1**+**14**+**27**+**7**. So what we will call intrinsic torsion are actually four tensors X_i, $i = 1, \dots, 4$ in these representations of G_2.

These objects are easier to calculate thanks to the following fact. For general G_2-structure manifolds the invariant form Φ is not covariantly constant in the Levi-Civita connection, and so $\nabla\Phi$ gives another measure of how far one is from having G_2 holonomy; in fact it is the same as intrinsic torsion [24]. In turn, all the information inside $\nabla\Phi$ are contained inside $d\Phi$ and $d * \Phi$. Decomposing these in G_2 representations gives us our X_i as

$$d\Phi = X_1(*\Phi) + X_4 \wedge \Phi + X_3\,, \qquad d(*\Phi) = \frac{4}{3}X_4 \wedge (*\Phi) + X_2 \wedge \Phi\,. \tag{2.3}$$

The first equation is a four-form and thus it contains **35**= **1**+**7**+**27**; the second is a five-form and so it decomposes as **21**=**7**+**14**. The **7** appear twice, but one can show that it is actually the same tensor up to a factor, as shown in (2.3). A further way we mention to compute torsions, which we illustrate briefly only in this example, is directly through the spinor equation. If one manages to put the right hand side of $D\vartheta = \dots$ in the form $K_{abc}q^{ab}\vartheta$ (using relations such as (3.9), which we will explain later), K_{abc} is already the torsion [25, 7]. In our case actually one can more efficiently put this right hand side in the form $(q_a + q_{ab}\gamma^b)\vartheta$, again using relations (3.9). Here q_{ab} can be a general tensor with two indices, which thus decomposes as **1**+**14**+**27**+**7** again and thus contains again all the information about intrinsic torsion. The **7** gets also contributions from q_a.

A similar story holds for SU(3) structures in six or seven dimensions. In six one has torsions in $(\mathbf{3}+\bar{\mathbf{3}}+\mathbf{1}) \otimes (\mathbf{3}+\bar{\mathbf{3}})$, which can be similarly as in the previous case encoded in dJ and $d\Omega$ [26, 27] (see also [6]). For seven dimensions the representations involved are even more, since we have $(2\times\mathbf{3}+2\times\bar{\mathbf{3}}+\mathbf{1}) \otimes (\mathbf{3}+\bar{\mathbf{3}}+\mathbf{1})$; again one can encode them in dJ, $d\Omega$ and dv, but this starts being of less practical use.

3. Supersymmetry constraints

The gravitino variation of eleven-dimensional supergravity in the presence of a nontrivial 4-form flux $G = dC$ and with vanishing gravitino background values reads

$$\delta\hat{\Psi}_A = \left\{\hat{D}_A[\hat{\omega}] + \frac{1}{144}\hat{G}_{BCDE}\left(\hat{\gamma}^{BCDE}{}_A - 8\hat{\gamma}^{CDE}\eta^B{}_A\right)\right\}\epsilon\,, \tag{3.1}$$

where ϵ is a Majorana spinor in eleven dimensions. Our conventions on indices here and in the following are

frame indices			coordinate indices		
$A, B, \dots$	$=$	$1, \dots, 11$	$M, N, \dots$	$=$	$1, \dots, 11$
$\alpha, \beta, \dots$	$=$	$1, \dots, 4$	$\mu, \nu, \dots$	$=$	$1, \dots, 4$
$a, b, \dots$	$=$	$5, \dots, 11$	$m, n, \dots$	$=$	$5, \dots, 11$

and hats refer to objects defined w.r.t. the eleven-dimensional frame. The signature of eleven-dimensional spacetime is $(-++\dots+)$ and the $\hat{\gamma}$-matrices satisfy $\{\hat{\gamma}_A, \hat{\gamma}_B\} = 2\eta_{AB}\mathbb{1}$.

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We want to consider warped compactifications

$$d\hat{s}_{11}^2 = e^{2\Delta} ds_4^2 + ds_7^2 \,, \tag{3.3}$$

where the warp factor depends only on the internal coordinates, $\Delta = \Delta(x^m)$ and where the four-dimensional spacetime with metric ds_4^2 is Minkowski. Lorentz invariance requires the background flux to be of the form

$$\begin{aligned} G &= 3\mu \frac{1}{4!}\epsilon_{\mu\nu\rho\sigma} dx^{\mu\nu\rho\sigma} + \frac{1}{4!} G_{mnpq} dx^{mnpq} \\ &= 3\mu \frac{1}{4!}\epsilon_{\alpha\beta\gamma\delta} e^{\alpha\beta\gamma\delta} + \frac{1}{4!} G_{abcd} e^{abcd} \\ &= 3\mu e^{-4\Delta} \frac{1}{4!}\epsilon_{\alpha\beta\gamma\delta} \hat{e}^{\alpha\beta\gamma\delta} + \frac{1}{4!} G_{abcd} \hat{e}^{abcd} \end{aligned} \tag{3.4}$$

with a real constant μ.

The decomposition for the $\hat{\gamma}$-matrices is the standard one,

$$\begin{aligned} \hat{\gamma}^\alpha &= \tilde{\gamma}^\alpha \otimes 1\!\!1 \qquad && \text{for} \quad \alpha = 1, \dots, 4 \\ \hat{\gamma}^a &= \tilde{\gamma}^{(5)} \otimes \gamma^a \qquad && \text{for} \quad a = 5, \dots, 11 \,, \end{aligned} \tag{3.5}$$

where $\tilde{\gamma}^{(5)} = i\tilde{\gamma}^1\tilde{\gamma}^2\tilde{\gamma}^3\tilde{\gamma}^4$ is the four-dimensional chirality operator. For explicit computations we will use the Majorana representation in which the γ-matrices are either real ($\tilde{\gamma}^\alpha$) or imaginary ($\tilde{\gamma}^{(5)}$ and γ^a). In this representation the Majorana condition on ϵ reduces to a reality constraint, $\epsilon^* = \epsilon$. Using now projectors $P_\pm \otimes \mathbb{I}$ we can write our class of ϵ as

$$\epsilon = \psi_+ \otimes \vartheta_+ + \psi_- \otimes \vartheta_- \tag{3.6}$$

again by four-dimensional invariance; also, $\vartheta_\pm$ depend only on the internal coordinates x^m. The four-dimensional spinors $\psi_\pm$ are (covariantly) constant

$$D_\alpha \psi_\pm = 0$$

and chiral, $\tilde{\gamma}^{(5)}\psi_\pm = \pm\psi_\pm$. The Majorana constraint on ϵ then requires $(\psi_\pm)^* = \psi_\mp$ and $(\vartheta_\pm)^* = \vartheta_\mp$. The gravitino variations (3.1) lead to the following supersymmetry constraints on the internal spinors,

$$0 = \left[\pm \left(\mu i e^{-4\Delta} + \frac{1}{2}(\partial_c \Delta)\gamma^c \right) + \frac{1}{144} G_{bcde}\gamma^{bcde} \right] \vartheta_\pm \tag{3.7a}$$

from the spacetime part $\alpha = 1, \dots, 4$ and

$$\begin{aligned} D_a[\omega]\vartheta_\pm &= \left[\frac{i}{2}\mu e^{-4\Delta}\gamma_a \mp \frac{1}{144} \left(G_{bcde}\gamma^{bcde}{}_a - 8G_{abcd}\gamma^{bcd} \right) \right] \vartheta_\pm, && (3.7\text{b}) \\ &= \left[\frac{i}{2}\mu e^{-4\Delta}\gamma_a \pm \left(\frac{i}{12}(*G)_{abc}\gamma^{bc} + \frac{1}{18} G_{abcd}\gamma^{bcd} \right) \right] \vartheta_\pm && (3.7\text{c}) \end{aligned}$$

from the internal part $a = 5, \dots, 11$, where we have defined $(*G)_{abc} \equiv \frac{1}{4!}\epsilon_{abcdefg}G^{defg}$.

3.1 Four-dimensional supersymmetry

A method to get equations from (3.7a) is simply to consider bilinear expressions

$$\vartheta_{\pm}\{G_{abcd}\gamma^{abcd}, \gamma_{a_1\ldots a_k}\}\vartheta_+ \tag{3.8}$$

(and the same with $\{\,,\,\} \to [\,,\,]$) and use (3.7a). It might a priori be non obvious which and how many of them generate all the possible relations. The spinor representation in seven dimensions however decomposes as $\mathbf{8}\to\mathbf{7}+\mathbf{1}\to\mathbf{3}+\bar{\mathbf{3}}+2\times\mathbf{1}$, which are $\gamma^a\vartheta_\pm$ and $\vartheta_\pm$. This simple fact suggests the answer: generating relations come from (3.8) for $k=0,1$. An equivalent but maybe more informative way of putting this — and which does not make use of the bilinears, whose structure we will determine later — is as follows.

The fact that the G_2 spinor is invariant implies, via the usual infinitesimal transformation for spinors $\delta\vartheta \sim q_{ab}\gamma^{ab}\vartheta$, that $\gamma^{ab}\vartheta$ belongs to the $\mathbf{7}$. Using the bilinear expression for Φ allows one to fix the constants as

$$\gamma_{ab}\vartheta = i\Phi_{abc}\gamma^c\vartheta$$

a relation well-known in the context of manifolds of G_2 *holonomy*, and which is in fact valid in general for manifolds of G_2 *structure*. Group theory then allows one to determine for every $\gamma^{a_1\cdots a_k}\vartheta$ which representations are present and which ones are not. For example, for $k=3$ we have that three-forms contain both $\mathbf{7}$ and $\mathbf{1}$, while for $k=5$ one only has $\mathbf{7}$. Coefficients can then be fixed by gamma matrix algebra and/or dualization. All this results in the relations

$$\begin{aligned}
\gamma_a\vartheta & & &\in \wedge^1_7\,,\\
\gamma_{ab}\vartheta &= i\Phi_{abc}\gamma^c\vartheta & &\in \wedge^2_7\,,\\
\gamma_{abc}\vartheta &= i\Phi_{abc}\vartheta-(*\Phi)_{abcd}\gamma^d\vartheta & &\in \wedge^3_1\oplus\wedge^3_7\,,\\
\gamma_{abcd}\vartheta &= (*\Phi)_{abcd}\vartheta-4i\Phi_{[abc}\gamma_{d]}\vartheta & &\in \wedge^4_1\oplus\wedge^4_7\,,\\
\gamma_{abcde}\vartheta &= 5(*\Phi)_{[abcd}\gamma_{e]}\vartheta & &\in \wedge^5_7\,,\\
\gamma_{abcdef}\vartheta &= i\epsilon_{abcdefg}\gamma^g\vartheta & &\in \wedge^6_7\,.
\end{aligned} \tag{3.9}$$

Here $\wedge^k_l$ denotes the irreducible G_2-representation of dimension l in the decomposition of $\wedge^k T^*Y$. Furthermore Φ_{abc} are the components of the G_2-invariant three-form Φ and are given by the structure constants of the imaginary octonions. When taking bilinears, most of the terms in (3.9) drop out due to $\vartheta^\dagger\gamma_a\vartheta=0$ leaving only the familiar $\Phi_{abc}=-i\vartheta^\dagger\gamma_{abc}\vartheta$ and $(*\Phi)_{abcd}=\vartheta^\dagger\gamma_{abcd}\vartheta$. We do emphasize however the importance of terms in $\wedge^3_7$ and $\wedge^4_7$ for our analysis. Similar but less pretty relations can be obtained for the SU(3) structure directly. In what follows it turns out sufficient to use just (3.9) and the explicit expression for the spinors $\vartheta_\pm$ (see 3.18)).

We define the following projections of the four-form flux

$$Q \equiv \frac{1}{4!}G_{abcd}(*\Phi)^{abcd}\,,\qquad Q_a \equiv \frac{1}{3!}G_{abcd}\Phi^{bcd}\,,\qquad Q_{ae}\equiv\frac{1}{3!}G_{abcd}(*\Phi)^{bcd}{}_e\,; \tag{3.10}$$

equivalently we can write

$$G_{abcd} = \frac{4}{7}Q(*\Phi)_{abcd}+Q_{[a}\Phi_{bcd]}-2\hat{Q}_{e[a}(*\Phi)^e{}_{bcd]}\,. \tag{3.11}$$

Q and Q_a only contain the projections onto the singlet and the **7** in the decomposition of the internal four-form flux into G_2-representations. $\hat{Q}_{ab} \equiv Q^{\mathbf{27}}_{ab}$ is symmetric and traceless. We note that all contractions of G with Φ and $*\Phi$ can be expressed solely in terms of Q, Q_a and $\hat{Q}_{ab}$. In particular, $Q_{ab} = -\frac{4}{7}Q\delta_{ab} + \frac{1}{2}\Phi_{abc}Q^c + \hat{Q}_{ab}$. Plugging (3.9) into (3.7a) and using the linear independence of ϑ and $\gamma^a\vartheta$, we obtain expressions of the form $(A + B_a\gamma^a)\vartheta$. Thus we see explicitly that A and B_a have to be put to zero and that this is all the information in the four dimensional equation. These are, writing real and imaginary parts separately,

$$\mu = 0\,, \qquad Q_a = 3\Phi_{abc}v^b\partial^c\Delta\,, \tag{3.12a}$$

$$Q_{ab}v^a v^b = 0\,, \qquad -3\partial_a\Delta = Qv_a + Q_{\{ab\}}v^b\,. \tag{3.12b}$$

This in particular sets to zero the Freund-Rubin parameter, which is not surprising as we are on Minkowski, and gives relations between G and $d\Delta$ on which we already commented in the introduction. We will find again some of these in a maybe more palatable form. From (3.12a) one can also derive

$$9(\partial_a\Delta\partial^a\Delta) = Q_a Q^a + Q^2\,, \tag{3.13}$$

from which one can see that in particular there cannot be any warped compactification if G has only components in the **27**. On the other side, (3.12b) implies that it's impossible to have G only in the **1** or only in the **7** either, since these are determined by $\hat{Q}_{ab}$. We recall [8] that when the warp factor is absent, all components of flux must vanish. So far we see that taking Δ to be constant gives $Q = Q_a = \hat{Q}_{ab}v^a = 0$. The rest of the conditions, like in [8] should come from analyzing the internal part. We will return to these points shortly in subsection 3.4.

3.2 Internal part

In order to study the constraints that (3.7b) impose on the geometry, we introduce the following bilinears,

$$\Xi_{a_1\ldots a_n} \equiv (\vartheta_+)^\dagger\gamma_{a_1\ldots a_n}\vartheta_+ \qquad \text{and} \qquad \tilde{\Xi}_{a_1\ldots a_n} \equiv (\vartheta_-)^\dagger\gamma_{a_1\ldots a_n}\vartheta_+ \tag{3.14}$$

and their associated forms

$$\Xi_n \equiv \frac{1}{n!}\Xi_{a_1\ldots a_n}e^{a_1\ldots a_n} \qquad \text{and} \qquad \tilde{\Xi}_n \equiv \frac{1}{n!}\tilde{\Xi}_{a_1\ldots a_n}e^{a_1\ldots a_n}\,. \tag{3.15}$$

The full set of bilinears for $n = 1, \ldots, 7$ is obviously redundant. One way to see relations between them would be to use Fierz identities. A faster way in this case is to use the expression for $\vartheta_\pm$ in terms of v and ϑ. But before we want to fix again possible ambiguities in that expression. We want to ask what are the normalizations $\vartheta_+^\dagger\vartheta_+ = \vartheta_-^\dagger\vartheta_- = \Xi$ and the scalar product $\vartheta_-^\dagger\vartheta_+ = \tilde{\Xi}$ (we have dropped the subscript 0 on the functions). For these we can derive differential equation using the usual methods, based on the supersymmetry constraints (3.7a) and (3.7b):

$$d\left(e^{-\Delta}\Xi\right) = 0, \qquad d\left(e^{2\Delta}\tilde{\Xi}\right) = 0\,. \tag{3.16}$$

To these one has to add another remark [2]. Note that since $\vartheta_\pm$ are by construction invariant spinors of an SU(3)-structure and $(\vartheta_\pm)^* = \vartheta_\mp$, there exists a connection w.r.t. which the normalized spinors $\frac{1}{\sqrt{\Xi}}\vartheta_\pm$ are covariantly constant. This implies that the scalar product between the normalized spinors

$$\frac{\vartheta_-^\dagger\vartheta_+}{\sqrt{\vartheta_-^\dagger\vartheta_-}\sqrt{\vartheta_+^\dagger\vartheta_+}} = \frac{\tilde{\Xi}}{\Xi} \tag{3.17}$$

has to be constant. For nontrivial warp factor $\Delta(x^m)$ this, combined with (3.16), forces $\tilde{\Xi}$ to vanish, so that the spinors have to be orthogonal and we find (1.1) again. Moreover, we can also substitute $\Xi = e^\Delta$ for the normalization. We write again the result for $\vartheta_\pm$:

$$\vartheta_\pm = \frac{1}{\sqrt{2}} e^{\frac{1}{2}\Delta \pm i\xi} \left(\mathbb{1} \pm v^a\gamma_a\right)\vartheta\,. \tag{3.18}$$

Note that the phase ξ reflects the existence of U(1) of SU(3) structures inside G_2 (see 2.2).

Using this we can now compute easily

$$\Xi \;= e^\Delta\,, \tag{3.19a}$$
$$\Xi_1 = e^\Delta v, \tag{3.19b}$$
$$\Xi_2 = ie^\Delta v \lrcorner \Phi\,, \tag{3.19c}$$
$$\Xi_3 = ie^\Delta v\wedge(v\lrcorner\Phi) = e^{-\Delta}\Xi_1\wedge\Xi_2\,, \tag{3.19d}$$
$$\tilde{\Xi}_3 = e^{\Delta+2i\xi}\left[i\Phi - iv\wedge(v\lrcorner\Phi) - v\lrcorner(*\Phi)\right] \tag{3.19e}$$
$$= e^{\Delta+2i\xi}\left[i\Phi - e^{-2\Delta}\Xi_1\wedge\Xi_2 - v\lrcorner(*\Phi)\right], \tag{3.19f}$$
$$\Xi_4 = e^\Delta\left[(*\Phi) - v\wedge(v\lrcorner(*\Phi))\right], \tag{3.19g}$$
$$\tilde{\Xi}_4 = e^{\Delta+2i\xi}\left[v\wedge(v\lrcorner(*\Phi)) - iv\wedge\Phi\right] = -e^{-\Delta}\Xi_1\wedge\tilde{\Xi}_3\,; \tag{3.19h}$$

the properties of these bilinears, as for example the vanishing of $\tilde{\Xi}_i$ for $i = 1,2$ and their duals, are consequence of properties of gamma matrices in seven dimensions. We see that a basis of generators are Ξ_1, Ξ_2 and $\tilde{\Xi}_3$. These are, up to normalization factors e^Δ, the invariant tensors which characterize our SU(3) structure in seven dimensions. What is now left, is to compute differential equations for these tensors using again the supersymmetry constraints (3.7a) and (3.7b). We actually do better and compute them for all the tensors we wrote:

$$d\left(e^\Delta\Xi_1\right) = 0\,, \qquad e^{-3\Delta}d\left(e^{3\Delta}\Xi_2\right) = -2i\,\Xi(*G) \tag{3.20a}$$

and

$$e^{-5\Delta}d\left(e^{5\Delta}\Xi_3\right) = 2i\,\Xi_1\wedge(*G)\,, \tag{3.20b}$$
$$e^{-2\Delta}d\left(e^{2\Delta}\tilde{\Xi}_3\right) = -2\tilde{\Xi}G\,, \tag{3.20c}$$
$$e^{-\Delta}d\left(e^\Delta\Xi_4\right) = 3G\wedge\Xi_1\,, \tag{3.20d}$$
$$e^{-4\Delta}d\left(e^{4\Delta}\tilde{\Xi}_4\right) = 0\,. \tag{3.20e}$$

As already pointed out, the set of Ξ_n and $\tilde{\Xi}_n$ is redundant and thus the systems (3.19) and (3.20) lead to extra consistency conditions. One might wonder whether these are new constraints to be added to (3.12). Due to the following argument, this can only be the case if we get relations involving G. A priori, one could have computed differential equations for (3.20a)–(3.20e) using only (3.7b) and not (3.7a). The result is a collection of rather cumbersome expressions involving partial contractions of G with Φ and $*\Phi$ (and no $d\Delta$). Thus any extra constraint in consistency conditions among (3.20a)–(3.20e), must be expressible in a form involving G only. This is not the case: the checks of consistency of (3.19) with (3.20) yield the single equation

$$6d\Delta \wedge \Xi_4 = -2i(*G) \wedge \Xi_2 - 3\,G \wedge \Xi_1\,. \tag{3.21}$$

Being a five-form, this equation can be split into **7** and **14** components. The vector part contains $d\Delta$, and thus it must be dependent on the conditions coming from (3.7a). The part in the **14** is instead purely in terms of G and is independent. One might want to consider this as a kind of "monopole equation" for the present class of compactifications. One can prove the following relations for a normalized vector v that are particularly useful in the checks mentioned above:

$$v \wedge (v \lrcorner \Phi) \wedge (v \lrcorner \Phi) = (v \lrcorner \Phi) \wedge \Phi\,, \tag{3.22a}$$
$$(v \lrcorner \Phi) \wedge (v \lrcorner (*\Phi)) = 0 \tag{3.22b}$$
$$(v \lrcorner \Phi) \wedge (v \lrcorner \Phi) = -2\,[(*\Phi) - v \wedge (v \lrcorner (*\Phi))]\,. \tag{3.22c}$$

The last relation implies in particular $\Xi_4 = \frac{e^{-\Delta}}{2}\Xi_2 \wedge \Xi_2$.

3.3 Intrinsic torsion

Note that the two- and three-form defining the SU(3)-structure (2.2) are given in terms of Ξ's as

$$J = -ie^{-\Delta}\Xi_2 \qquad \text{and} \qquad \psi_3 = \mathrm{Re}\left\{e^{-\Delta}\tilde{\Xi}_3\right\}, \tag{3.23}$$

whereas the three- and four-form of the G_2-structure have representations,

$$\begin{aligned} \Phi &= \mathrm{Im}(e^{-\Delta-2i\xi}\tilde{\Xi}_3) - ie^{-2\Delta}\Xi_1 \wedge \Xi_2\,, \\ (*\Phi) &= e^{-\Delta}\Xi_4 - e^{-2\Delta}\Xi_1 \wedge \mathrm{Re}(e^{-2i\xi}\tilde{\Xi}_3)\,. \end{aligned} \tag{3.24}$$

The equations (1.3) were written in terms of the maybe more familiar complex three-form Ω, which in terms of the above reads $\Omega \equiv \psi_3 + iv \lrcorner (*\psi_3)$.

We can finally compute G_2 intrinsic torsions as promised. For this we need

$$d\Phi = -3d\Delta \wedge [\Phi + v \wedge (v \lrcorner \Phi)] + 2d\xi \wedge [v \lrcorner (*\Phi)] + 2v \wedge (*G)\,, \tag{3.25a}$$
$$d(*\Phi) = -2d\Delta \wedge (*\Phi) - 3d\Delta \wedge v \wedge [v \lrcorner (*\Phi)] - 2d\xi \wedge v \wedge \Phi + 3G \wedge v\,. \tag{3.25b}$$

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Projecting these into representations we get (up to overall factors)

$$X_{\mathbf{1}} = \partial_a \xi v^a \,, \tag{3.26a}$$

$$(X_{\mathbf{14}})_{ab} = \frac{1}{2}\Phi_{abef} v^e Q^f + 2v_{[a}Q_{b]} - \frac{1}{2}\Phi_{abc}\hat{Q}^c{}_e v^e + \hat{Q}^c{}_{[a}\Phi_{b]ec}v^e \,, \tag{3.26b}$$

$$(X_{\mathbf{7}})_a = -8\,\Phi_{abc}v^b\partial^c\xi + 3\hat{Q}_{ab}v^b - 15\partial_a\Delta + \frac{65}{7}Qv_a \,, \tag{3.26c}$$

$$(X_{\mathbf{27}})_{ab} = -\,4\left(\partial_{\{a}\xi v_{b\}} - \frac{1}{7}\delta_{ab}(\partial_c\xi v^c)\right) - Q_{\{a}v_{b\}} + \hat{Q}_{\{a}{}^e\Phi_{b\}ec}v^c \,. \tag{3.26d}$$

Here we denoted torsions by representations. Thanks to (3.12a), (3.12b), one can also derive

$$v_a = \frac{1}{\partial_e\Delta\partial^e\Delta}\left[6\Phi_{abc}Q^b\partial^c\Delta - \frac{Q}{3}\partial_a\Delta\right] \tag{3.27}$$

to eliminate v and make (3.26a)–(3.26d) purely in terms of physical quantities. Note also that in all these expressions Q and Q_a can also be eliminated in favor of $\hat{Q}_{ab}$.

Of course in a sense these expressions do not mean that one can forget about v altogether: one still has to check the differential equation for v,

$$d(e^{2\Delta}v) = 0 \,, \tag{3.28}$$

separately. But we have decoupled dv from $d\Phi$. About (3.28) we can actually say more: it also has a mathematical meaning. It is known that this implies [25, 2, 1] that the seven-dimensional metric can be written in a product form

$$ds^2_7 = ds^2_6 + e^{-4\Delta}dx^2_7$$

with no restriction, however, on the coordinate dependence of Δ and ds^2_6. Indeed v needs not be Killing: the symmetric part of its covariant derivative, which we have not written above, reads

$$D_{\{a}\Xi_{b\}} = \frac{1}{3}Q\delta_{ab} - Q_{\{ab\}} \,. \tag{3.29}$$

So if we want this to be a Killing vector, we have to impose that G is in the **7** only. But, as already noticed, from (3.12b) one sees that if Q and $Q_{\{ab\}}$ vanish, the warp factor Δ is constant. In fact it is not hard to see that the equation (3.29) implies the second equation in (3.12b).

3.4 A short summary

We have presented here a set of general relations between the components of intrinsic torsion on a generic seven-dimensional manifold, admitting spinors and thus a G_2 structure, and the components of four-form flux. Due to the existence of a full classification of manifolds admitting G_2 structure, one may hope that a similar classification of M-theory backgrounds can be achieved. We would like to emphasize that the results presented here are just a set of necessary conditions for preserving supersymmetry. Namely given a manifold with a particular set of intrinsic torsions, we know now what is the possible profile of

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the four-form flux needed for preserving supersymmetry, and vice versa. While the existence of the SU(3) structure is crucial for preserving $\mathcal{N}=1$ supersymmetry, as explained above on *any* seven-dimensional spin manifold this structure is already present due to existence of (two!) nowhere vanishing vector field(s). Supersymmetry does however impose a differential equation on this vector field (an analogue of the Killing vector equation). A general analysis of existence of solutions for this equation might be an interesting problem, which is beyond the scope of this paper.

It is time now to collect all the information concerning the four-form flux. While the relations between the components of the flux and intrinsic torsion is in general complicated, we see that supersymmetry imposes strong constraints on components of the flux. In particular, the "primitive" part, namely $G_{\mathbf{27}}$ is the most important part of the flux, and determines the two other components, through the expressions

$$Q=\frac{7}{4}\hat{Q}_{ab}v^a v^b\,,\qquad Q_a=-2\Phi_{abc}v^b\hat{Q}^c{}_d v^d\,. \tag{3.30}$$

It is easy to see that vanishing of **27** leads to vanishing of the other two components, but not the other way around. Thus, $G_{\mathbf{27}}$ cannot be zero. From other side, due to (3.13), it cannot be the only component turned on. Thus in order to have a warped compactification, primitivity is not enough.

Going back to the case of constant warp factor, one can see that to the conditions $Q=Q_a=\hat{Q}_{ab}v^a=0$, stated above, equation (3.21) adds $\hat{Q}^c{}_{[a}\Phi_{b]ec}v^e=0$. This still does not eliminate $\hat{Q}_{ab}$, and thus $G_{\mathbf{27}}$, entirely, and one has to go back to the integrability conditions [8]. For $\Delta=0$, the integrability is certainly the most restrictive, since the Ricci scalar is negative semi-definite and the equation $-R+G^2=0$ forces both the vanishing of flux and Ricci-flateness. This condition is much less restrictive (and less useful) for a non-constant warp factor, since now the Ricci scalar is no longer semi-definite and the equation acquires new terms like $\Phi^{abc}D_aX_{bc}$ which are not positive-definite. In other words, for a warped product involving a generic seven-manifold with components of intrinsic torsion X, after having built $G_{\mathbf{1}}$ and $G_{\mathbf{7}}$ in terms of the primitive flux according to (3.30), one has enough freedom to satisfy (3.21) for a non-trivial four-form $G_{\mathbf{27}}$.

4. SU(2) structures and towards $\mathcal{N}=2$.

We already saw that the existence on the internal manifold of vector fields without zeros can have rather far-reaching consequences for supersymmetry. We have concentrated so far on the "minimal" case of SU(3) structure with $\mathcal{N}=1$ supersymmetry where only one such vector was actively involved. As we saw the SU(3) structure comes out rather naturally from the seven-dimensional parts $\vartheta_\pm$ of $\frac{1}{2}(1\pm\gamma^{(5)})\otimes\mathbb{1}\epsilon$.

However since we have a pair of vector fields on seven-manifolds and thus, as discussed in section 2, SU(2) structure, it is natural to ask what the consequences of this are for supersymmetry. Of course, one could easily add the second vector field in (3.18) by $v\to\sum a_i v^i$, but it is not hard to see that this replacement preserves as much supersymmetry as the original Ansatz. Moreover since we have argued that (3.18) is the most general possibility, by field redefinitions we can bring the new spinor to this form again. To be short, using SU(2) structure to preserve $\mathcal{N}=1$ supersymmetry does not give anything new in comparison to SU(3) case.

The situation will be different when one will want to look for $\mathcal{N}=2$ solutions where one can use the SU(2) structure in a more interesting fashion. Here we make the first step in that direction by writing down the generalization of (3.18) suitable for four-dimensional $\mathcal{N}=2$.

Strictly speaking one would only need three spinors to define an SU(2) structure. But we can easily come up with a fourth one, which does not add extra structure, but is more compatible with four-dimensional chirality. We will now have four spinors $\vartheta^i_\pm$, $i=1,2$. The three spinors of SU(2) structure can be thought of as ϑ, $v_1^a\gamma_a\vartheta$ and $v_2^a\gamma_a\vartheta$, where we now make use of both the vectors we discussed in section 2. Then the fourth spinor can be easily constructed as the Clifford action of both v_i on ϑ, that is $\tilde{\vartheta}\equiv v_1^a v_2^b\gamma^{ab}\vartheta$. If one wants, this too can be cast in a form similar to the other spinors writing it as $\tilde{\vartheta}=v_3^a\gamma_a\vartheta$, where $v_3^a\equiv i\Phi^a{}_{bc}v_2^b v_3^c$.

We can now combine these four spinors in a way similar to (3.18), to produce the four spinors we want $\vartheta^i_\pm$ $(i=1,2)$. We still want them to be orthogonal to each other for arguments similar to those leading to (3.18); the U(1) freedom that we had before (the phase $e^{i\xi}$) gets now replaced by a U(2) freedom. So one can start from whatever choice and act with a U(2) matrix on it. A possibility to express this is, very explicitly,

$$\vartheta^1_+ = e^{i\xi}(a+bv_1+av_2+bv_3)\cdot\vartheta \tag{4.1a}$$

$$\vartheta^2_+ = e^{i\xi}(-\bar{b}+\bar{a}v_1-\bar{b}v_2+\bar{a}v_3)\cdot\vartheta\,, \tag{4.1b}$$

where we have now denoted the Clifford multiplication by a dot, $v\cdot\vartheta\equiv v_a\gamma^a\vartheta$; the ϑ^i_- are then $\vartheta^i_-=(\vartheta^i_+)^*$, and one has to remember that v_3 is purely imaginary. The eleven-dimensional spinor now is $\epsilon=\psi^i_+\otimes\vartheta^i_+ + \psi^i_-\otimes\vartheta^i_-$.

Thus indeed the existence of SU(2) structure on seven-manifolds leads to possibility of preserving $\mathcal{N}=2$ supersymmetry. The possibility of starting with a certain background and enhancing supersymmetry by adjusting the fluxes looks interesting and to our opinion deserves further study.

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References

[1] J.P. Gauntlett, D. Martelli, S. Pakis and D. Waldram, *G-structures and wrapped NS5-branes*, `hep-th/0205050`.

[2] J.P. Gauntlett, D. Martelli and D. Waldram, *Superstrings with intrinsic torsion*, hep-th/0302158.

[3] S. Gurrieri, J. Louis, A. Micu and D. Waldram, *Mirror symmetry in generalized Calabi-Yau compactifications*, *Nucl. Phys.* **B 654** (2003) 61 [hep-th/0211102].

[4] G.L. Cardoso, G. Curio, G. Dall'Agata, D. Lust, P. Manousselis and G. Zoupanos, *Non-kaehler string backgrounds and their five torsion classes*, *Nucl. Phys.* **B 652** (2003) 5 [hep-th/0211118].

[5] P. Kaste, R. Minasian, M. Petrini and A. Tomasiello, *Kaluza-Klein bundles and manifolds of exceptional holonomy*, *J. High Energy Phys.* **09** (2002) 033 [hep-th/0206213].

[6] P. Kaste, R. Minasian, M. Petrini and A. Tomasiello, *Nontrivial RR two-form field strength and* SU(3)*-structure*, *Fortschr. Phys.* **51** (2003) 764 [hep-th/0301063].

[7] J.P. Gauntlett and S. Pakis, *The geometry of $D = 11$ Killing spinors*, *J. High Energy Phys.* **04** (2003) 039 [hep-th/0212008].

[8] P. Candelas and D.J. Raine, *Spontaneous compactification and supersymmetry in $D = 11$ supergravity*, *Nucl. Phys.* **B 248** (1984) 415.

[9] B. de Wit, H. Nicolai and N.P. Warner, *The embedding of gauged $N = 8$ supergravity into $D = 11$ supergravity*, *Nucl. Phys.* **B 255** (1985) 29.

[10] S. Gukov, *Solitons, superpotentials and calibrations*, *Nucl. Phys.* **B 574** (2000) 169 [hep-th/9911011].

[11] B.S. Acharya and B. Spence, *Flux, supersymmetry and M-theory on 7-manifolds*, hep-th/0007213.

[12] K. Becker and M. Becker, *Compactifying M-theory to four dimensions*, *J. High Energy Phys.* **11** (2000) 029 [hep-th/0010282].

[13] T. Ali, *M-theory on seven manifolds with G-fluxes*, hep-th/0111220.

[14] K. Behrndt and C. Jeschek, *Fluxes in M-theory on 7-manifolds and G structures*, *J. High Energy Phys.* **04** (2003) 002 [hep-th/0302047].

[15] B. Brinne, A. Fayyazuddin, S. Mukhopadhyay and D.J. Smith, *Supergravity M5-branes wrapped on riemann surfaces and their QFT duals*, *J. High Energy Phys.* **12** (2000) 013 [hep-th/0009047];
B. Brinne, A. Fayyazuddin, T.Z. Husain and D.J. Smith, *$N = 1$ M5-brane geometries*, *J. High Energy Phys.* **03** (2001) 052 [hep-th/0012194].

[16] M. Cvetič, H. Lu and C.N. Pope, *Massless 3-branes in M-theory*, *Nucl. Phys.* **B 613** (2001) 167 [hep-th/0105096];
M. Cvetič, G.W. Gibbons, H. Lu and C.N. Pope, *Supersymmetric M3-branes and G_2 manifolds*, *Nucl. Phys.* **B 620** (2002) 3 [hep-th/0106026].

[17] R. Minasian and D. Tsimpis, *Hopf reductions, fluxes and branes*, *Nucl. Phys.* **B 613** (2001) 127 [hep-th/0106266].

[18] J.M. Maldacena and C. Núñez, *Supergravity description of field theories on curved manifolds and a no go theorem*, *Int. J. Mod. Phys.* **A 16** (2001) 822 [hep-th/0007018].

[19] B. de Wit, D.J. Smit and N.D. Hari Dass, *Residual supersymmetry of compactified $D = 10$ supergravity*, *Nucl. Phys.* **B 283** (1987) 165.

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[20] Th. Friedrich, I. Kath, A. Moroianu and U. Semmelmann, *On nearly parallel G_2-structures*, *J. Geom. Phys.* **23** (1997) 259.

[21] K. Becker and M. Becker, *M-theory on eight-manifolds*, *Nucl. Phys.* **B 477** (1996) 155 [hep-th/9605053].

[22] D. Joyce, *Compact manifolds with special holonomy*, Oxford University Press, Oxford 2000.

[23] E. Thomas, *Vector fields on manifolds*, Bull. Amer. Math. Soc. **75** (1969) 643.

[24] M. Fernandez and A. Gray, *Riemannian manifolds with structure group G_2*, *Ann. di Mat. Pura ed Appl.* **32** (1982) 19.

[25] S. Salamon, *Riemannian geometry and holonomy groups*, *Pitsman Res. Notes in Math.* **201**, Longman, Harlow 1989.

[26] A. Gray and L. Hervella, *The sixteen classes of almost hermitian manifolds and their linear invariant*, *Ann. di Mat. Pura ed Appl.* (IV) **123** (1980) 35.

[27] S. Chiossi and S. Salamon, *The intrinsic torsion of* SU(3) *and G_2 structures*, in *Proc. conf. Differential Geometry Valencia 2001*, math.DG/0202282.